Journal of Statistical Physics, Vol. 56, Nos. 1/2, 1989

Tangential Flow in Fluid Membranes. Absence of Renormalization Effects

H. Kleinert[1]

Received September 14, 1988

The effect of the tangential flow, in fluid membranes, on the renormalization of the curvature elastic constant κ is studied and it is shown that the softening of κ when averaged over increasingly short distance scales is the same as for ideal surfaces carrying no material. This is in contrast with a recent claim by Förster. The physical and formal differences in the two treatments are pinpointed.

KEY WORDS: Statistical mechanics; membranes; functional integral; path integral surfaces.

In a recent note, Ami and Kleinert[(1),2] showed that the short-wavelength fluctuations in membranes with arbitrary elastic constants μ, λ lead to the same softening of the extrinsic curvature stiffness as in ideal membranes[(3)] in which the material particles are allowed to rearrange themselves freely within the surface. The short-distance renormalization is given by

$$\kappa = \kappa_0 - (c/2)(1/4\pi)\ln(q_{\max}^2/q_{\min}^2) \tag{1}$$

with $c = 3$, where $q_{\max}^{-1}$ is the short-distance cutoff and $q_{\min}^{-1}$ the longest distance over which the fluctuations have been integrated out. The number $c = 3$ has by now been obtained in various ways.[(4,5)] It disagrees with Helfrich's original[(6)] and final result,[(7)] which is $c = 1$.

In a recent note, however, Förster[(8)] argues that the almost incompressible fluid nature of the membrane material would lead to Helfrich's

[1] Institut für Theoretische Physik, Freie Universität Berlin, 1000 Berlin 33, Germany.

[2] Recently, it has been argued that the infrared divergences caused by elasticity renormalize the curvature energy at long wavelength.[(2)] This effect is unrelated to the question discussed here and does not change the argument. In fact, it is an effect linked in an essential way to shear elasticity and vanishes for a fluid membrane.

0022-4715/89/0700-0227$06.00/0

number $c=3-2=1$ after all. Actually, there is a sign error in the additional term found by Förster, so that, if his arguments were valid, his value of c would really be $c=3+2=5$. This trivial error will, however, not be of concern here, but rather the theoretical basis of his calculation. Since c enters directly into various observable quantities, such as the size distribution of spherical vesicles,[9,10] it is important to know its precise value.

Förster takes recourse to the most natural way of constructing the measure of a classical path integral, based on the canonical formalism of the time-dependent problem. The path integral is simply the product of the integrals over all canonical conjugate variables at each time. For a membrane which is a perfect fluid along the surface $\mathbf{x}(\xi, t)$ and performs small fluctuations around a smooth background configuration $\mathbf{x}_0(\xi)$ which is in stress equilibrium and has a uniform mass density ρ_0, the total kinetic energy reads

$$E_{\text{kin}} = (\rho_0/2) \int d^2\xi \; g_0^{1/2}(\dot{v}^2 + \dot{\tau}_i \cdot \dot{\tau}^i) \tag{2}$$

where g_{0ij} is the background metric $\partial_i \mathbf{x}_0 \, \partial_j \mathbf{x}_0$, g_0 is its determinant, and v and τ^i are the normal and tangential displacements. The free part of the action is

$$\mathscr{A}_0 = \int dt \left\{ -i \int d^2\xi \; g_0^{1/2}(\dot{v} p_v + \dot{\tau}^i p_{\tau^i}) \right.$$
$$\left. + \; (1/\rho_0) \int d^2\xi \; g_0^{1/2}(p_v^2/2 + p_\tau^2/2) \right\} \tag{3}$$

and the quantum mechanical path integral is to be taken with a measure

$$\int D\mu = \prod_t \prod_\xi \left\{ \int dv \int d^2\tau^i \int (dp_v/2\pi) \int [d^2 p_\tau/(2\pi)^2] \right\} \tag{4}$$

where $\prod_\xi$ runs over some infinitely fine grated parametrization lattice and $\prod_t$ over a grated time axis, continued to imaginary values $t \to -it$ for quantum statistics. As far as the present problem is concerned, we may write the extrinsic curvature part of the energy for small displacements from $\mathbf{x}_0(\xi)$, including the one-loop corrections, effectively as follows[10]:

$$E_{\text{curv}} = (\kappa/2) \int d^2\xi \; g_0^{1/2}[(D^2 v)^2 + (3/2)\, C_0^2 v D^2 v] \tag{5}$$

where the covariant derivative D is done in the background metric g_0 and C_{0ij} is the extrinsic curvature matrix assumed to be almost constant (using the notation $C_0 \equiv C_{0i}{}^i$). The reparametrization invariance causes τ^i to drop out.

In the original calculations for an ideal membrane, the path integral over v led to a fluctuation energy

$$\begin{aligned} &T/2 \operatorname{tr} \ln(\delta^2 E_{\text{curv}}/\delta v\, \delta v) \\ &\quad = (T/2) \operatorname{tr} \ln[D^4 + (3/2)\, C_0^2 D^2] \\ &\quad \approx (T/2) \operatorname{tr} \ln D^4 + (T/2) \operatorname{tr}[-(3/2)\, C_0(1/-D^2)\, C_0] \end{aligned} \tag{6}$$

and thus directly to the thermal softening law (1).

Let us now look at the possible changes brought about by the elastic properties within the surface. The elastic energy reads[1]

$$E_{\text{el}} = \int d^2\xi\, g_0^{1/2}[\mu(u_i{}^j - \tfrac{1}{2}\delta_i{}^j u_l{}^l)^2 + (K/2)\, u_l{}^{l2}] \tag{7}$$

where μ, K are the elastic constants and

$$u_{ij} = D_i \tau_j + D_j \tau_i - 2C_{0ij} v + D_i v\, D_j v + \cdots \tag{8}$$

is the strain tensor, obtained by expanding the metric g_{ij} around the equilibrium background configuration ($u^j{}_i \equiv g_{ik}\, g_0{}^{kj} - \delta^j{}_i$). For our purpose, we only have to keep the first three linear terms τ^i and v. The full quantum partition function is given by the path integral

$$Z = \int D\mu \exp\left\{ -\mathscr{A}_0 - \int dt\, (E_{\text{cirv}} + E_{\text{el}}) \right\} \tag{9}$$

The thermal partition function is obtained via the classical limit, doing the integrals (5) for time-independent functions only, thereby dropping all kinetic pieces in $\mathscr{A}_0$ and obtaining an overall factor $1/T$. This gives

$$\begin{aligned} Z_{\text{cl}} = \prod_\xi \left\{ \int dv \int D^2\tau^i \int (dp_v/2\pi) \int [d^2 p_\tau/(2\pi)^2] \right\} \\ \times \exp\left\{ (1/T)\left[-(1/2\rho_0) \int d^2\xi\, g_0^{1/2}(p_v^2/2 + p_\tau^2/2) - E_{\text{curv}} - E_{\text{el}} \right] \right\} \end{aligned} \tag{10}$$

after which the p integrations lead to

$$Z_{\text{cl}} = \prod_\xi \left[\int dv^4\, g_0^{1/2} \int d^2\tau^i g_0 \right] \exp[-(1/T)(E_{\text{curv}} + E_{\text{el}})] \tag{11}$$

This was the path integral used in ref. 1 to show that the elastic properties do not change the short-distance renormalization of the ideal membrane result in Eq. (1).

Let us now turn to Förster's argument. He takes the incompressibility limit $K \to \infty$ so that (7) leads to a δ-function constraint for u_l^l enforcing $D_i \tau^i = C_0 v$. He chooses the rotation-free solution

$$\tau^i = (1/D^2)\, D^i C_0 v \tag{12}$$

inserts this into the kinetic energy (3), and arrives at a kinetic energy for v fluctuations

$$E_{\text{kin}} = (\rho_0/2) \int d^2\xi \; g_0^{1/2} \{\dot{v}^2 + \dot{v} C_0 [1/(-D^2)]\, C_0 \dot{v}\} \tag{13}$$

where the time derivatives of C_0 and D^2 vanish for a static background configuration. Assuming a smooth background, he is allowed to ignore space derivatives of C_0. The further discussion is then based on the action

$$\mathscr{A}_0 = \int dt \left\{ \int d^2\xi \; g_0^{1/2} [-i\dot{v} p_v + (1/2\rho_0)\, p_v A^{-1} p_v] + E_{\text{curv}} \right\} \tag{14}$$

where A is the functional matrix

$$A(\xi, \xi') = \delta^2(\xi - \xi') + C_0(\xi)(1/-D^2)(\xi, \xi')\, C_0(\xi') \tag{15}$$

Going to the classical limit and integrating out p_v gives the effective measure

$$\prod_\xi \left(\int dv \int dp_v/2\pi \right)$$

$$= \prod_\xi \left(\int dv \; g_0^{1/4} \right) \exp\{(-1/2)\, \text{tr} \ln[1 + C_0 (1/-D^2)\, C_0]\} \tag{16}$$

Expanding the trace log increases the 3/2 in Eq. (6) to 5/2, thus leading to the number $c = 5$ in (1), as a consequence of the incompressibility of the membrane.

What is wrong with this argument? For a comparison, let us go to the classical limit starting from the proper quantum partition function (10). For now, we shall ignore the shear distortions. Integrating out the canonical momenta and doing a quadratic completion in the τ^i variables involving $\dot{\tau}^2$ and $K(D_i \tau^i - C_0 v)^2$, we see that the kinetic piece of v becomes

$$(\rho_0/2) \int dt \; \{d^2\xi \; g_0^{1/2} \dot{v} [1 - v^2 C_0 (\partial_t^2 + v^2 D^2)^{-1} C_0]\, \dot{v}\} \tag{17}$$

where $v = (K/\rho_0)^{1/2}$ is the velocity of compressional sound waves within the surface. The operator between the v's can be rewritten as $\dot{v}B\dot{v}$, where B is a functional matrix

$$B = 1 - v^2 C_0 (\partial_t^2 + v^2 D^2)^{-1} C_0 \tag{18}$$

Beside this, the integral over τ^i yields the entropy for the sound waves, which is a purely intrinsic quantity and influences only the Gaussian curvature energy. So it is of no concern here. Adding to (18) the extrinsic curvature energy and doing the path integral gives the partition function

$$\exp\left(-(1/2)\sum_{n=0}^{\infty} \mathrm{tr}_\xi \ln\{\omega_n^2[1 + v^2 C_0(\omega_n^2 - v^2 D^2)^{-1} C_0] + (D^2)^2 + (3/2)\, C_0^2 D^2\} + (1/2)\sum_{n=1}^{\infty} \mathrm{tr}_\xi \ln \omega_n^2\right) \tag{19}$$

where ω_n are the Matsubara frequencies $2\pi n T$ and the tr_ξ is taken only with respect to the ξ variables. For high temperatures, all frequencies $n \neq 0$ are so large that the remainder in the trace log of (19) is irrelevant.[10] The $n = 0$ term gives the classical partition function which is the same as (8), thus confirming the previous result of ref. 1.

The place where Förster's treatment deviates from this canonical procedure can now easily be pinpointed. His aim is to impose the incompressibility limit from the outset. Within the canonical treatment, this amounts to taking the limit $v \to \infty$ in all ω_n^2 terms for fixed n, so that they become $\omega_n^2[1 + C_0(-D)^{-2} C_0]$. We see immediately that this is not permissible. No matter how large v, the ω_n^2 will eventually exceed $-v^2 D^2$ and it is only due to this feature that the sum converges. In order to enforce a convergence after all, Förster had to modify the path integral by a factor $\det[1 + C_0(-D)^{-2} C_0]$, which he did with his extra term in (15) and which led to his result (16). Thus, we see that his result is based on an extreme unphysical incompressibility limit. In this context, we should note that even though one speaks of a membrane as an almost incompressible two-dimensional viscous fluid, K is still small enough to justify the classical limit at room temperature (i.e., its Debye temperature lies below room temperature). Indeed, with the typical modulus of compression, $K \approx 450$ dyn/cm, even for a low density $\rho_0 \approx 100\, m_p/\text{Å}^2$ ($m_p \equiv$ atomic mass unit), the sound velocity is $v \approx 200$ m/sec and the thermal energy Tk_B at room temperature, 3×10^{-14} erg, is much larger than the energy of the shortest possible waves $q_{max} \approx 2\pi/\text{Å}$, which is $vq_{max} \approx 2 \times 10^{-15}$ erg. This shows that an extreme incompressibility limit makes no physical sense.

REFERENCES

1. S. Ami and H. Kleinert, *Phys. Lett. A* **120**:207 (1987).
2. D. Nelson and L. Peliti, *J. Phys.* (Paris) **48**:1085 (1987); F. David, E. Guitter, and L. Peliti, *J. Phys.* (Paris) **48**:2059 (1987).
3. L. Peliti and S. Leibler, *Phys. Rev. Lett.* **54**:1960 (1985).
4. H. Kleinert, *Phys. Lett.* **114A**:263 (1986); **B174:**335 (1986).
5. D. Förster, *Phys. Lett.* **114A**:115 (1986).
6. W. Helfrich, *J. Phys.* (Paris) **46**:1263 (1985).
7. W. Helfrich, *J. Phys.* (Paris) (in press).
8. D. Förster, *Europhys. Lett.* **4**:65 (1987).
9. W. Helfrich, *J. Phys.* (Paris) **47**:321 (1986).
10. H. Kleinert, *Phys. Lett. A* **116**:57 (1986).

Communicated by J. L. Lebowitz